No.

Date 11 · 4 · 2019 Curvilinear Laplacian, divergence, curl

From Math notebook [4/10/2017] : Geometry (differential)

$$d\vec{r} = \frac{d\vec{r}}{d\xi^i} d\xi^i = \left|\frac{d\vec{r}}{d\xi^i}\right| e_{\xi^i} d\xi^i = h_{\xi^i} e_{\xi^i} d\xi^i$$

$$\nabla f = \frac{1}{h_{\xi^i}} \frac{\partial f}{\partial \xi^i} e_{\xi^i}$$

$$\therefore \nabla \xi^i = \frac{e_{\xi^i}}{h_{\xi^i}}$$

$$\nabla \times \nabla \xi_i = 0 = \nabla \times \frac{e_{\xi^i}}{h_{\xi^i}}$$

assuming orthogonal coordinates,

$$e_{\xi^i} \times e_{\xi^j} = e_{\xi^k} \quad , i \neq j \neq k$$

$$\therefore \nabla \cdot \left[e_{\xi^k} / (h_{\xi^i} h_{\xi^j}) \right] = \nabla \cdot \left(\frac{e_{\xi^i} \times e_{\xi^j}}{h_{\xi^i} h_{\xi^j}} \right) = \nabla \cdot (\nabla \xi^i \times \nabla \xi^j)$$

$$= \nabla \xi^j \cdot (\nabla \times \nabla \xi^i) - \nabla \xi^i \cdot (\nabla \times \nabla \xi^j)$$

$$= 0$$

$$\therefore \nabla \cdot \vec{V} = \nabla \cdot \left[\frac{e_{\xi^1}}{h_{\xi^2} h_{\xi^3}} (h_{\xi^2} h_{\xi^3} V^1) + \frac{e_{\xi^2}}{h_{\xi^1} h_{\xi^3}} (h_{\xi^1} h_{\xi^3} V^2) + \frac{e_{\xi^3}}{h_{\xi^1} h_{\xi^2}} (h_{\xi^1} h_{\xi^2} V^3) \right]$$

$$= \frac{e_{\xi^1}}{h_{\xi^2} h_{\xi^3}} \cdot \nabla (h_{\xi^2} h_{\xi^3} V^1) + \frac{e_{\xi^2}}{h_{\xi^1} h_{\xi^3}} \cdot \nabla (h_{\xi^1} h_{\xi^3} V^2) + \frac{e_{\xi^3}}{h_{\xi^1} h_{\xi^2}} \cdot \nabla (h_{\xi^1} h_{\xi^2} V^3)$$

Curvilinear Laplacian, curl, divergence

Date 11 . 4 . 2019

$\cancel{e_{\xi_1}(\nabla\cdot\vec{V})}$ $\because e_{\xi_1}\cdot\nabla(h_{\xi_2}h_{\xi_3}V^1) = \frac{1}{h_{\xi_1}}\frac{\partial}{\partial\xi_1}(h_{\xi_2}h_{\xi_3}V^1)$, etc ...

$$\boxed{\begin{aligned}\therefore\ \nabla\cdot\vec{V} &= \frac{1}{h_{\xi_1}h_{\xi_2}h_{\xi_3}}\left[\frac{\partial}{\partial\xi_1}(h_{\xi_2}h_{\xi_3}V^1)+\frac{\partial}{\partial\xi_2}(h_{\xi_1}h_{\xi_3}V^2)+\frac{\partial}{\partial\xi_3}(h_{\xi_1}h_{\xi_2}V^3)\right]\\ \therefore\ \nabla^2\vec{V} &= \frac{1}{h_{\xi_1}h_{\xi_2}h_{\xi_3}}\left[\frac{\partial}{\partial\xi_1}\left(\frac{h_{\xi_2}h_{\xi_3}}{h_{\xi_1}}\frac{\partial V^1}{\partial\xi_1}\right)+\frac{\partial}{\partial\xi_2}\left(\frac{h_{\xi_1}h_{\xi_3}}{h_{\xi_2}}\frac{\partial V^2}{\partial\xi_2}\right)+\frac{\partial}{\partial\xi_3}\left(\frac{h_{\xi_1}h_{\xi_2}}{h_{\xi_3}}\frac{\partial V^3}{\partial\xi_3}\right)\right]\end{aligned}}$$

Since $\nabla\times\frac{e_{\xi_i}}{h_{\xi_i}} = 0$

$$\nabla\times\vec{V} = \nabla\times\left[\frac{e_{\xi_1}}{h_{\xi_1}}h_{\xi_1}V^1+\frac{e_{\xi_2}}{h_{\xi_2}}h_{\xi_2}V^2+\frac{e_{\xi_3}}{h_{\xi_3}}h_{\xi_3}V^3\right]$$

$$= -\frac{e_{\xi_1}}{h_{\xi_1}}\times\nabla(h_{\xi_1}V^1) - \frac{e_{\xi_2}}{h_{\xi_2}}\times\nabla(h_{\xi_2}V^2) + \frac{e_{\xi_3}}{h_{\xi_3}}\times\nabla(h_{\xi_3}V^3)$$

$$\because\ \nabla\times(\phi\vec{W}) = \phi(\nabla\times\vec{W}) - \vec{W}\times\nabla\phi$$

$$\because\ \frac{e_{\xi_1}}{h_{\xi_1}}\times\nabla(h_{\xi_1}V^1) = \frac{e_{\xi_1}}{h_{\xi_1}}\times\left(\frac{e_{\xi_1}}{h_{\xi_1}}\frac{\partial V^1}{\partial\xi_1}+\frac{e_{\xi_2}}{h_{\xi_2}}\frac{\partial V^1}{\partial\xi_2}+\frac{e_{\xi_3}}{h_{\xi_3}}\frac{\partial V^1}{\partial\xi_3}\right)$$

$$= \frac{e_{\xi_3}}{h_{\xi_1}h_{\xi_2}}\frac{\partial V^1}{\partial\xi_2} - \frac{e_{\xi_2}}{h_{\xi_1}h_{\xi_3}}\frac{\partial V^1}{\partial\xi_3}$$

$$\therefore\ \nabla\times\vec{V} = -\frac{1}{h_{\xi_1}h_{\xi_2}h_{\xi_3}}\Big[e_{\xi_3}h_{\xi_3}\frac{\partial V^1}{\partial\xi_2} - e_{\xi_2}h_{\xi_2}\frac{\partial V^1}{\partial\xi_3} - e_{\xi_3}h_{\xi_3}\frac{\partial V^2}{\partial\xi_1}$$
$$+ e_{\xi_1}h_{\xi_1}\frac{\partial V^2}{\partial\xi_3} + e_{\xi_2}h_{\xi_2}\frac{\partial V^3}{\partial\xi_1} - e_{\xi_1}h_{\xi_1}\frac{\partial V^3}{\partial\xi_2}\Big]$$

$$\boxed{\nabla\times\vec{V} = \frac{1}{h_{\xi_1}h_{\xi_2}h_{\xi_3}}\begin{vmatrix} h_{\xi_1}e_{\xi_1} & h_{\xi_2}e_{\xi_2} & h_{\xi_3}e_{\xi_3}\\ \frac{\partial}{\partial\xi_1} & \frac{\partial}{\partial\xi_2} & \frac{\partial}{\partial\xi_3}\\ h_{\xi_1}V^1 & h_{\xi_2}V^2 & h_{\xi_3}V^3\end{vmatrix}}$$